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A model of nonlocal potential and alpha decay of deformed even nuclei

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Abstract. The problem of α decay of *deformed* even nuclei is treated here by assuming the barrier to be the usual *anisotropic* electrostatic potential superimposed by a *nonlocal* α–nuclear potential. The method employed is similar to that developed in a recently published work on α angular distributions. Accordingly, the electrostatic potential is, for simplicity, described by the usual quadrupole deformation parameter $\beta_{2,0}$, while the nonlocal part is taken for the same reason to be spheroidal, described by a deformation parameter ϵ_l. The latter potential being momentum-dependent, ϵ_l may naturally take different values for different α angular momenta l. Furthermore, the barrier penetration effects are taken to be the main factors governing hindrances to the excited state transitions. With these assumptions a formula for hindrance factor, $(\mathrm{H.F.})_l$, is derived.

Then from straightforward calculations, and with a *single* value of ϵ_8 for different nuclei ($\beta_{2,0}$ being known from experiments), the values of $(\mathrm{H.F.})_8$ are found to be within *fifty per cent* of the empirical values, whereas the values from the shell-model theory or from a purely Coulomb or static barrier hypothesis show large discrepancies. Similarly, for $l \leqslant 6$, overall agreement for an average ϵ_l is better than obtained previously. On the other hand, it is remarkable that for the best-fit values of an ϵ_l varying slightly with nuclear mass A and charge Z (which appear not unreasonable by analogy with the similar variations of $\beta_{2,0}$) one obtains *exact* agreement with (within ten per cent) *all* the empirical $(\mathrm{H.F.})_l$ values to be discussed here.

1. Introduction

The intensities of α fine structures for heavy elements, when calculated on the basis of the simple barrier penetration theory, are found to be systematically greater than the observed values by very large factors (up to 15 000) for higher α angular momenta l (cf. Perlman and Rasmussen 1957). After the discovery of the rotational level structures in the daughter nucleus (Bohr and Mottelson 1953, Perlman and Asaro 1954) the theories of α decay of deformed nuclei were developed in a number of rapidly succeeding papers (cf. Hanna 1959) by taking into account various effects of deformations of the nuclear surface as well as of α angular momentum l. It was then generally concluded that if the barrier is assumed to be purely Coulombic and if the nucleus is axially symmetric, and also if the α wavefunction on the nuclear surface is a constant, then the agreement between the theory and experiment is not satisfactory. For example, in his comprehensive treatment, Fröman (1957) has shown that with chosen values of $\beta_{2,0}$—which are, however, different from empirical values—the variational character of b_l with Z is not theoretically reproduced. Gol'din *et al.* (1958) have included higher multipole moments in their theory and also discussed the difficulties involved in some of these methods.

On the other hand, Rostovskii (1961) has discussed the problem by assuming the nucleus to be nonaxial (cf. Davydov and Filipov 1958), a treatment unlike that in the above papers.

But he finds that with the chosen values of the parameter γ, which measures deviations from axial symmetry, the calculated intensities still largely differ from the experimental data except for the transitions to 2^+ states.

More recently an extensive study of this problem has been made by Mang (1957), Mang and Rasmussen (1962) and Poggenburg *et al.* (1969). Their basic formulation is to calculate the α cluster probability, taken to be the main factors governing the hindrances to excited states, in terms of the shell-model product wave-functions of the constituent nucleons. A simpler treatment which also has a bearing on the alpha formation and nuclear shell model was given by Brussard and Tolhoek (1958). As for the potential barrier the Coulomb potential was superimposed by an external α–nuclear potential. But, as discussed elsewhere (and below), there are reasons to believe that the said α–nuclear potential is nonlocal in character. In Mang's theory this has been assumed to be *static*. It must be stated that there are no indications from the shell-model theory to support a *static* α–nuclear potential. Also, a static barrier with shell-model effects is inadequate, as may be seen from columns 7 and 10 of table 6. Wilkinson (1961) has questioned the shell-model theory taking the calculations by Harada (1961) to show that the calculated values are too low by factors of 10^2 to 10^4, depending on the detailed assumptions made. Besides, it has also been noted that the *conclusions of the shell-model theory are critically dependent on the choices of the potential barrier.* Hence the need arises for reviewing the situation.

An alternative approach to the α decay problem would therefore be to assume that the α cluster probability is not largely dependent on the α angular momentum. This assumption is not new. The earliest work in which the α decay reduced widths δ_l^2 for excited states are taken to be the same as for the ground state transitions is the theory by Bethe (1937). In many subsequent works this valuable concept was used (cf. the review article by Rasmussen 1965). Obviously if we assume $\delta_l^2 \simeq \delta_0^2$ then the explanation of the hindrance factors will have to be sought in terms of the variations of the only other factor in the formula for the decay constant λ—namely the penetrability factor P. In calculating P the assumption that a purely electrostatic barrier is an oversimplification and that this should be superimposed by an external α–nuclear potential is now established (cf. any recent paper on α decay). But certain points about the character of this α–nuclear interaction, although mentioned more than once in our previous papers, may be recalled. It has been known for a long time (Van Vleck 1935, Fay 1936, Bardeen 1937, and see the review article by Glassgold 1958) that the average potential felt by a nucleon is dependent on its velocity. Thus the statistical nuclear model with exchange forces gives an interaction between a nucleon and its neighbours of a nonlocal type in coordinate space. The energy dependence of the nuclear interaction is also evidenced from a number of experiments (cf. Weisskopf 1957) and from the observed variations of the optical parameters, which are however not uniquely known, in the α scattering experiments (cf. for example Igo and Thaler 1957, El Nadi and Riad 1965, McFadden and Satchler 1966 and Thompson *et al.* 1967).

A phenomenological interaction of nonlocal type has been used by Frahn (1956) and other authors to account for the modified propagation character of nucleons within nuclear matter and the said velocity dependence of the optical parameters in α–nuclear scattering. An equivalent interpretation due to Wheeler (1936) is to assign a reduced effective mass to the interacting nucleon; he used this method to account for nuclear saturation. The nucleon–nucleon interaction has also been discussed, for example, by Levy (1952) and by Levy and Marshak (1955), in terms of a nonlocal operator in the meson theory of nuclear forces.

In the present context we are not directly concerned with the question as to the equivalence of the above formally different approaches to the many-body character of the nuclear interaction. What we intended to emphasize is that if the interaction between a nucleon and its close neighbours is *not* independent of the state of the interacting system then this should be true in the α decay process also. Of course, if the effect of such a state-dependent α–nuclear potential is not negligible then this must be capable of explaining some of the observed characteristics of α decay. Most important of them is the finding that the hindrances to the excited state transitions increase systematically with increasing α angular momentum l. We shall show here that this is a natural consequence of the l dependence of the said nonlocal barrier penetration probability.

From the above considerations it was proposed (Chaudhury 1960) that the nucleons in the two separated bodies (namely, the α particle and the daughter nucleus) interact in a way which may be represented by an overall nonlocal α–nuclear potential. The justification for an overall potential is provided by the considerations based mainly on the works of Brueckner and Levinson (1954, 1955, also cf. Bethe 1956) on the many-body problem of nuclear structure.

One significant difference between the scattering problem and the α decay process is that the parameters of the optical potential chosen to fit the scattering data involve bombarding energies quite high compared with the α decay energy. Also, the motion of a nucleon in the nuclear matter is different from that of an α particle in the peripheral region of the daughter nucleus. Therefore, in choosing the appropriate form of the nonlocal interaction kernel, one has to depend on the trial method. Initially we assumed this to be the same as that used in the effective mass approximation. This form of nonlocality amounts to replacing the reduced mass of the α particle by the variable effective mass $\mu(r)$, in the Gamow factor, which, however, is found to lead to too large a change in the barrier penetration probability (cf. Chaudhury 1963). It thus appeared that the momentum dependence of the α–nuclear potential was different from that associated with the effective mass approximation. This point was also noted by other authors (W. E. Frahn private communication, see also Preston 1963). Hence in a recent paper (Chaudhury 1966) a simpler isotropic nonlocal potential was defined in terms of which it was possible to explain (agreements being in most cases within 25 per cent of the observed values) the relative intensities of α spectra, particularly the parity-unfavoured transitions (not previously explained) for *spherical* nuclei with charge number $Z \leqslant 92$ and neutron number $N \leqslant 138$. More recently this spherically symmetric nonlocal potential was suitably extended (Chaudhury 1970 to be referred to as I) to take into account the effects of deformations of the nuclear surface, and is found useful in interpreting the observed anisotropy of the α emissions from oriented ^{237}Np nuclei. In view of these successes, and considering the experimental basis, it is reasonable that the problem of α decay of *deformed* nuclei in the region $A > 226$ and $N > 138$ be treated along these lines. We shall, however, restrict ourselves here to the even parity α transitions in the even–even nuclei, to avoid further complications.

In § 2 we set up relevant wave equations by taking into account the above considerations regarding the potential barrier. For simplicity, we confine our attention to quadrupole shapes of the nuclear surface so that the electrostatic part is described by the parameter $\beta_{2,0}$. To be consistent, the nonlocal part of the barrier will be described by a spheroidal deformation parameter, say ϵ_l. In our calculations, the values of $\beta_{2,0}$ will be taken from the list of values by Bell *et al.* (1960), from the measurements of half-lives of excited states in different nuclei. We might as well take the somewhat different values of $\beta_{2,0}$ given by Rester *et al.* (1961) obtained from the spectra of internal conversion electrons from the Coulomb excitations of heavy elements. But the data available by the latter method are limited. Again, since the α–nuclear potential itself is momentum-dependent, its deformation parameter ϵ_l may naturally take different values for different l. All the other parameters involved are known fairly accurately either from previous experiments or from theoretical considerations (see § 4). Furthermore, as already mentioned, the main factors governing hindrances to the excited state transitions will be discussed here in terms of the above-mentioned barrier penetration probability, and it appears that for the present problem the assumption of large variations with l of the α cluster probability is not necessary.†

Approximate solutions of the wave equations are obtained by following a three-dimensional WKB method due to Christy (1955) and adapted to the present problem. In this way the desired formula for the α decay constant and hence the formula for the hindrance factor is obtained. In the next section the results of calculations are presented for the average ϵ_l, as well as for its best-fit values, and compared with the experimental data.

From straightforward calculations it is found that with a *single* value of ϵ_8 for different nuclei the values of $(\mathrm{H.F.})_8$ are within *fifty per cent* of the empirical values, whereas the shell-model theory and calculations on the basis of a static or the purely Coulomb barrier hypothesis differ largely from the empirical values. For $l = 6$ and with a *single* value of ϵ_6 for different nuclei, the same order of agreements are obtained except for the unusual case of $^{230}\mathrm{Th}$ (with $(\mathrm{H.F.})_6 = 13\,000$) for which ϵ_6 is to be chosen somewhat smaller. For $l \leqslant 4$, the effects of momentum dependence of the barrier are not expected to be as predominant and the dependence of ϵ_l on A and Z is found to be more pronounced. It is however remarkable that for the best-fit values of *an* ϵ_l varying slightly with A and Z (which is not unreasonable, by analogy with the similar variations of $\beta_{2,0}$) one obtains *exact* agreements with (within ten per cent) *all* the empirical $(\mathrm{H.F.})_l$ values to be discussed here.

Nevertheless, it should not be forgotten that there are other important problems, such as the effects of α cluster probability, multipole moments and those due to nonaxial shapes of the nuclei. However, the barrier effects, particularly for higher α angular momentum l, are clearly predominant; also, if one intends to study the nonlocal effects, it is necessary to restrict the number of unknown parameters to a minimum and hence we do not consider them here.

2. Wave equations and their solutions

The solutions of the Schrödinger equation for the deformed nonlocal barrier discussed above are given in I. We shall here briefly indicate the method of arriving at these solutions.

It has been shown in I that the equation for the radial part of the partial α wave

$$\psi(\boldsymbol{r}) = r^{-1}U_l(r)Y_{l,m}(\theta,\varphi)$$

† This condition was also found valid for the α decay of spherical nuclei (cf. the relation $F_l \simeq F_0$ in Chaudhury 1966 p. 184).

for the *isotropic* barrier takes the form

$$\frac{\hbar^2}{2\mu}\left(U_l''(r)-\frac{l(l+1)}{r^2}U_l(r)\right)+\{E_l-u_0(r)-v_l^{(0)}(r)\}$$
$$\times\left(1+\frac{b^2\mu}{2\hbar^2}v_0^{(2)}(r)\right)U_l(r) = b\pi^{-1/2}v_0^{(1)}(r)\,U_l'(r) \tag{1}$$

where primes denote differentiation with respect to r, μ is the reduced mass, $u_0(r) = 2(Z-2)e^2/r$ (Z being the charge number of the parent nucleus), e is the electronic charge, and the *isotropic* nonlocal energy is of the general form

$$v_l^{(n)}(r) = V(r)\,p_l^{(n)}(r) \tag{2}$$

where $V(r)$ is the static part assumed to be given by the optical model potential; the purely nonlocal part of order n (shown as superscript) is given by

$$p_l^{(n)}(r) = g^{(n)}(z)\left\{1-\frac{l(l+1)}{4r^2}\left(1-\frac{g^{(2)}(z)}{g^{(0)}(z)}\right)\right\} \tag{2.1}$$

where $z = (r-R_1)/b$, R_1 is the *inner* turning point and b is the range of nonlocality, and

$$g^{(0)}(z) = \tfrac{1}{2}\{1+2\pi^{-1/2}\int \exp(-z^2)\,\mathrm{d}z\}$$

$$g^{(1)}(z) = \tfrac{1}{2}\{1-2\int \exp(-z^2)z\,\mathrm{d}z\}$$

$$g^{(2)}(z) = \tfrac{1}{2}\{1+4\pi^{-1/2}\int \exp(-z^2)z^2\,\mathrm{d}z\}. \tag{2.2}$$

The right hand side of equation (1) is small—as $g^{(1)}(z)$ vanishes except at the immediate neighbourhood of R_1—and will be ignored.

Now for deformed nuclei both $u_0(r)$ and $v_l^{(n)}(r)$ should be replaced by the corresponding angle-dependent terms $u(\boldsymbol{r}')$ and $v_l^{(n)}(\boldsymbol{r}')$, referred to the body-fixed system of coordinates $\boldsymbol{r}'(= r, \theta', \varphi')$ with the same origin as for the laboratory system of coordinates, $\boldsymbol{r}(= r, \theta, \varphi)$. As is well-known for quadrupole deformations,

$$u(\boldsymbol{r}') = u_0(r)+u_2(r)\beta_{2,0}Y_{2,0}(\theta') \tag{3}$$

where

$$u_2(r) = \tfrac{3}{5}u_0(r)\,(R_c/r)^2 \tag{3.1}$$

and $R_c(= r_0(A-4)^{1/3}\times 10^{-13}$ cm) is the radius of a hypothetical nuclear sphere and is connected to the spheroidal surface $R(\theta')$ by the relation

$$R(\theta') = R_c\{1+\beta_{2,0}Y_{2,0}(\theta')\}. \tag{3.2}$$

On the other hand, for the axially symmetric nonlocal potential we find that (cf. paper I)

$$v_l^{(n)}(\boldsymbol{r}') = v_l^{(n)}(r)-\bar{\epsilon}\{rp_l^{(n)}V'(r)+rV(r)\,p_l^{(n)\prime}\}Y_{2,0}(\theta')$$
$$+(5/8\pi)\epsilon_l^2\{rp_l^{(n)}(r)\;V'(r)+rV(r)\,p_l^{(n)\prime}(r)\} \tag{4}$$

where $\bar{\epsilon} = \{\epsilon_l+(\tfrac{5}{4\pi})^{1/2}\epsilon_l^2\}$. Collecting these terms, the equation for the deformed barrier corresponding to equation (1) becomes

$$\Delta\psi = F_l^2\psi \tag{5}$$

where

$$F_l^2 = \frac{2\mu}{\hbar^2}\left(u(\boldsymbol{r}')+v_l^{(0)}(\boldsymbol{r}')+\frac{b^2\mu}{2\hbar^2}v_0^{(2)}(\boldsymbol{r}')\{u(\boldsymbol{r}')+v_l^{(0)}(\boldsymbol{r}')-E_l\}-E_l\right). \tag{5.1}$$

Now the procedure to solve equation (5) differs significantly from that generally used (cf. Davidson 1965) in the case of the electrostatic barrier for which there is no actual inner turning point. On the other hand, the existence of such a turning point requires that the extremal equation should be taken as

$$\int_{r_{\mathrm{P}'}}^{R_{\mathrm{a}}} F_l\,\mathrm{d}r + \int_{R_{\mathrm{a}}}^{r_{\mathrm{P}}} F_l\,\mathrm{d}r + \ldots = \text{minimum} \tag{6}$$

where P′ is a point on the spheroidal nuclear surface $R(\theta')$ (cf. equation (3.2)) and P is somewhere within the barrier region. In the present context the lower limit in equation (6) should nowhere be less than the *inner* turning point (which in three dimensions will now be denoted as $R_{\mathrm{i}}(\theta')$ instead of R_{i}—cf. below equation (2.1)). This is because F_l^2 in the integrals changes from a positive value outside $R_{\mathrm{i}}(\theta')$ to a negative value inside it, which is not allowed.

Now, eccentricity σ being small, it can easily be shown that

$$R_{\mathrm{i}}(\theta') = R_{\mathrm{a}}\{(1-\tfrac{1}{3}\sigma^2)+\tfrac{1}{3}\sigma^2P_{2,0}(\cos\theta')\}. \tag{7}$$

Since the resultant deformation depends now on both $\beta_{2,0}$ and ϵ_l, we introduce for convenience, by analogy with equation (3.2), an 'effective' deformation parameter $\bar{\alpha}$, such that

$$R_{\mathrm{i}}(\theta') = \bar{R}\{1+\bar{\alpha}Y_{2,0}(\theta')\}. \tag{7.1}$$

To our approximation, retaining terms up to the order b^2 and ϵ_l^2, we get from equation (5.1):

$$F_l(r,\theta') = \{(2\mu)^{1/2}/\hbar\}\{\mathscr{S}_l(r)+\mathscr{T}_l(r,\theta')\} \tag{8}$$

where

$$\mathscr{S}_l(r) = (u_{\mathrm{eff}}-E_l)^{1/2}. \tag{8.1}$$

The effective potential u_{eff} now replaces the previous isotropic Coulomb potential $u_0(r)$ and

$$u_{\mathrm{eff}} = \{u_0(r)-v_l^{(0)}(r)-\Delta v_l\} \tag{8.2}$$

where

$$\Delta v_l = (b^2\mu/2\hbar^2)\,p_0^{(2)}V(r)\,\{u_0(r)-V(r)\,p_l^{(0)}(r)-E_l\}. \tag{8.3}$$

Again the angle-dependent part in equation (8) after simplification reduces to

$$\begin{aligned}\mathscr{T}_l(r,\theta') = {} & (1/2\mathscr{S}_l)[\{u_2(r)\beta_{2,0}-W_2(r)\bar{\epsilon}-(\Delta K_1-\Delta K_2)\}Y_{2,0}(\theta') \\ & +W_2(r)5\epsilon_l^2/8\pi]\end{aligned} \tag{8.4}$$

where

$$W_2(r) = V(r)\{\bar{r}p_l^{(0)}-rp_l^{(0)\prime}(r)\} \tag{8.5}$$

$$\begin{aligned}\Delta K_1 = {} & (b^2\mu/2\hbar^2)\,p_0^{(2)}(r)V(r)\,[u_2(r)\beta_{2,0}+\bar{r}\epsilon_l\{u_0(r)-E_l\}-\epsilon_lV(r) \\ & \times\{2\bar{r}p_l^{(0)}(r)-rp_l^{(0)\prime}(r)\}]\end{aligned} \tag{8.6}$$

$$\Delta K_2 = \{(b^2\mu/2\hbar^2)rV(r)p_0^{(2)\prime}(r)\epsilon_l\}\{u_0(r)-V(r)p_l^{(0)}-E_l\} \tag{8.7}$$

with $\bar{r} = r \times 10^{13}/0{\cdot}574$. Now the expansion coefficients in the state vector depend on a number of quantum numbers in addition to l and, instead of the $U_l(r)$ of equation (1), let this function be denoted in the body-fixed and space-fixed systems of coordinates by $S^{j_i}_{l,m,K_f}(r)$ and by $B^{j_i}_{l,\nu,K_f}(r)$ respectively. The subscripts i and f indicate the initial and final states, l, m and ν denote the α angular momentum and its projections on the z axis (lab. system) and on the nuclear axis of symmetry, j, M and K are the total angular momentum of the nucleus and its projections on the two axes respectively. By using the rotation matrix the connection between S and B is obtainable as

$$S^{j_i}_{l,m,K_f}(r) = \sum_{\nu} (-1)^{j_f - j_i + \nu}(j_i,\, l,\, K_f + \nu,\, -\nu | j_f,\, K_f) B^{j_i}_{l,\nu,K_f}(r). \tag{9}$$

Now we apply the mean-value theorem in evaluating the integral

$$\int_{R_i(\theta')}^{R_a} \mathscr{S}_l(r)\, \mathrm{d}r = \{1 - K(\zeta')\}^{1/2} \int_{R_i(\theta')}^{R_a} \{u_0(r) - E_l\}^{1/2}\, \mathrm{d}r \tag{10}$$

where

$$K(\zeta') = [\{v_l^{(0)}(\zeta') + \Delta v_l(\zeta')\}/\{u_0(\zeta') - E_l\}] \tag{10.1}$$

with $R_i(\theta') \leqslant \zeta' \leqslant R_a$. Since the minimum of $R_i(\theta')$ and R_a are nearly the same, for calculation purposes we can set $K(\zeta') = K(R_a)$ in equation (10.1) without appreciable error.

Now, using equations (10), (10.1) and (8) in (5), and after simplification, we obtain the solution

$$B^{j_i}_{l,\nu,K_f}(r) = R_a\left(\frac{U_l(E_l, r)}{U_l(E_l, R_a)}\right) \sum_{l'} h_{l',\nu} q_{l,l'}^{\nu}(a_l, \zeta_l) \tag{11}$$

with the boundary condition that the α wave function $\psi_0(\theta, \varphi)$ on the nuclear surface is expanded as

$$\psi_0(\theta, \varphi) = \sum_{l'} \sum_{\nu'} h_{l',\nu'} Y_{l',\nu'}(\theta, \varphi) \tag{12}$$

where the modified matrix

$$q_{l,l'}^{\nu}(a_l, \zeta_l) = \int_0^{\pi} \Theta_{l,\nu} \exp[-\{\zeta_l + a_l P_{2,0}(\theta)\}]\theta_{l',\nu} \sin\theta\, \mathrm{d}\theta \tag{13}$$

and

$$a_l = \{(\tfrac{5}{4\pi})^{1/2} \mathscr{I}_l + (\mathscr{I}_2 - \zeta_l)\} \tag{13.1}$$

$$\mathscr{I}_1 = (\sqrt{2\mu}/\hbar) \int_{R_a}^{R_0} \Big[[u_2(r)\beta_{2,0} - V(r)\bar{\epsilon}\{\bar{r}p_l^{(0)}(r) - rp_l^{(0)\prime}(r)\} + \Delta K_1 - \Delta K_2]/2$$

$$\times \{u_0(r) - v_l^{(0)} - \Delta v_l - E_l\}^{1/2}\Big]\, \mathrm{d}r \tag{13.3}$$

$$\mathscr{I}_2 = \frac{(2\mu)^{1/2}}{\hbar} \int_{R_a}^{R_0} \frac{V(r)\bar{\epsilon}\{\bar{r}p_l^{(0)}(r) - rp_l^{(0)\prime}(r)\}}{2\{u_0(r) - v_l^{(0)}(r) - \Delta v_l - E_l\}^{1/2}}\, \mathrm{d}r \tag{13.4}$$

$$\zeta_l = \tfrac{1}{3}\sigma^2\chi[(kR_a/\chi)\{1 - (3kR_a/2\chi)\}(1 - K_a)]^{1/2} + \mathscr{I}_2 \tag{13.5}$$

where

$$\chi = 2(Z-2)e^2(2\mu)^{1/2}/\hbar E_l^{1/2} \qquad k = (2\mu E_l)^{1/2}/\hbar \tag{13.6}$$

and $K_a = K(R_a)$. Now U_l occurring in equation (11) is obtained by solving equation (1) by the usual method of approximation as

$$U_l(E_l, R_a) = U_0(E_l, R_a) \exp\left(\frac{y_0 l(l+1)}{\chi} I_0\right) \tag{14}$$

where

$$y_0 = \{(Z-2)e^2/E_l^{1/2}\} \tag{14.1}$$

and

$$I_0 = \int_{R_a}^{R_0} \frac{1+(\mu b^2/2\hbar^2)V(r)\{g^{(0)} - p_0^{(2)}(r)\}}{r^2\{u_0(r) - V(r)g^{(0)} - E_l - \Delta v_0\}^{1/2}} \mathrm{d}r \tag{14.2}$$

where R_0 is the outer turning point, that is, $2(Z-2)e^2/E_l$.

3. Alpha-decay constant

The α decay probability, with angular momentum l and for the transition from the initial state j_i, K_i to the state j_f, K_f, is, by definition,

$$\lambda_{K_f,l,j_f} = \operatorname*{limit}_{r\to\infty} 4\pi v_l |S^{j_i}_{l,m,K_f}(r)|^2 \tag{15}$$

where the S function is given by equations (9) and (11).

From what has been said in § 1 about the α cluster probability we can set $h_{l',\nu} \ll h_{0,0}$ in equation (11) and also $\nu = 0$ for even–even nuclear transitions. Hence, neglecting higher order terms, the formula for the hindrance factor with α energy normalized for $\lambda_{0,0}$ is finally obtained as

$$(\mathrm{H.F.})_l = \left|\frac{q_{0,0}(\zeta_0, a_0)}{q_{l,0}(\zeta_l, a_l)}\right|^2 \exp\left(\frac{2l(l+1)}{\chi}(y_0 I_0)\right). \tag{16}$$

In the next section we give the results of calculations of hindrance factors from equation (16). Here we may mention that equation (16) contains the centrifugal term and $q_{l,0}$, both of which involve exponential functions of l-dependent nonlocal parameters. It is therefore expected that the results are quite sensitive to the deformation parameter ϵ_l, particularly in the cases for large values of l (see figure 3).

4. Results

To evaluate hindrance factors from equation (16) it may be seen that the integrals I_0 occurring in equation (14.2) and $\mathscr{I}_1$ and $\mathscr{I}_2$ (cf. equations (13.3) and (13.4)) which occur in ζ_l and a_l are not analytically known. We therefore calculate these integrals by taking advantage of a computer and by using a modified Simpson's rule discussed in I. In the present calculations the number of strips of width d for the entire integral range R_a to R_0 is taken to be 100. On the other hand, for the quantities $g^{(0)}(z)$, that is, Erf(z) and $p_0^{(2)}$ defined in equations (2.2) and (2.1) and involved in the integrals I_0, $\mathscr{I}_1$ and $\mathscr{I}_2$, one obtains sufficiently accurate values with only 60 strips for every value of z.

Now of the different parameters involved—namely, the range of nonlocality b, the hypothetical nuclear radius R_c (cf. equation (3.2)), the inner turning point $R_i(\theta')$ (equation (7.1)), the eccentricity σ (cf. below equation (7)), the outer turning point R_0, the quadrupole deformation parameter β_2 and the nonlocality parameter ϵ_l—only the last one is unknown. The other parameters are known fairly accurately either from experiments or from theoretical considerations. In I we found that the nonlocality b is in the range $0{\cdot}5 \leqslant b \leqslant 0{\cdot}9$ fm and the penetration factor is not very sensitive to the values of b within this range. We shall, therefore, take $b = 0{\cdot}7$ fm. In the definition of $R_c(= r_0 \times (A-4)^{1/3} \times 10^{-13}$ cm) there is some ambiguity in the choice of r_0. In α decay theories r_0 is generally indicated as $r_0 \simeq 1{\cdot}44$, whereas smaller values are obtained from other sources (cf. Hofstadter 1956). In the present calculations it is found that r_0 may be taken as small as $1{\cdot}20$. Next, the inner turning point $R_i(\theta')$ is obtained for different θ' by solving the equation $F_l{}^2 = 0$ (see equation (8)) by the iterative method and by neglecting small terms such as $W_2(r)(5/8\pi)\epsilon_l{}^2$ and those involving products of $\beta_2 b^2$ and $\epsilon_l b^2$. This makes the values of $R_i(\theta')$ slightly different from the exact values. But it can be shown that the difference is not more than one strip-width d and may be ignored. We thus obtain the values of $R_i(\theta')$ for $\theta' = 0$ or π, $\cos^{-1}(1/3)^{1/2}$ and for $\theta' = \pi/2$, giving R_a, $\bar{R}$ and R_{min} respectively.

The eccentricity σ is then easily obtained from R_a and R_{min}. It may be mentioned that the empirical hindrance factors are obtained by taking the nuclear radius for the ground state transitions to be the same as for the excited states. Hence to be consistent we have calculated $R_i(\theta')$ by setting $l = 0$ for the different α energies E_l in a fine-structure pattern for a given nucleus. The outer turning point R_0 is of course given by $2(Z-2)e^2/E_l$. Furthermore, as already mentioned in § 1, β_2 will be taken here from the list of empirical values (Bell *et al.* 1960) except for ^{246}Cf, ^{250}Cf and

Table 1

Parent nucleus	α-particle energy, (l) E_l(MeV)	Empirical β_2	Nonlocal parameter ϵ_l	Inner turning point $R_i(\theta)$ (fm) R_a	$\bar{R}$	R_{min}	Eccentricity σ (equation (7))	Effective deformation $\bar{\alpha}$ (equation (7·1))
^{238}Pu	4·996 (8)	0·251	0·350	9·187	9·059	8·974	0·214	0·0224
	5·195 (6)		0·280	9·154	9·064	9·006	0·179	0·0157
	5·352 (4)		0·160	9·095	9·069	9·051	0·098	0·0045
	5·452 (2)		0·200	9·118	9·072	9·041	0·130	0·0081
^{242}Cm	5·596 (8)	0·271	0·390	9·245	9·099	9·005	0·226	0·0254
	5·806 (6)		0·280	9·196	9·105	9·055	0·174	0·0157
	5·964 (4)		0·135	9·123	9·109	9·102	0·067	0·0023
	6·066 (2)		0·200	9·160	9·112	9·090	0·123	0·0080
^{230}Th	4·436 (6)	0·197	0·165	9·029	8·984	8·965	0·119	0·0079
	4·471 (4)		0·220	9·059	8·985	8·948	0·156	0·0130
	4·615 (2)		0·220	9·063	8·989	8·953	0·156	0·0130
^{236}Pu	5·442 (6)	0·257	0·250	9·120	9·048	9·002	0·161	0·0126
^{240}Pu	4·851 (6)	0·263	0·300	9·177	9·078	9·013	0·188	0·0172
^{244}Cm	5·511 (6)	0·278	0·260	9·194	9·119	9·077	0·159	0·0129
	5·658 (4)		0·120	9·131	9·124	9·119	0·050	0·0013
^{246}Cf	6·462 (6)	(0·280)	0·290	9·240	9·151	9·094	0·177	0·0153
	6·613 (4)		0·160	9·179	9·156	9·142	0·089	0·0041
	6·711 (2)		0·175	9·190	9·159	9·140	0·104	0·0054
^{228}Th	5·208 (4)	0·195	0·240	9·067	8·989	8·939	0·167	0·0137
^{230}U	5·658 (4)	0·220	0·250	9·086	9·002	8·958	0·167	0·0148
^{234}U	4·598 (4)	0·233	0·240	9·091	9·019	8·973	0·161	0·0128
^{238}U	4·020 (4)	0·233	0·200	9·101	9·049	9·016	0·137	0·0092
	4·147 (2)		0·210	9·110	9·053	9·016	0·143	0·0100
^{250}Cf	5·980 (2)	(0·282)	0·160	9·203	9·183	9·169	0·086	0·0034
^{254}Fm	7·064 (4)	(0·288)	0·180	9·268	9·236	9·217	0·104	0·0055
	7·158 (2)		0·150	9·254	9·238	9·229	0·073	0·0027

^{254}Fm for which β_2 are not known. We have chosen the values shown in parentheses in column 3 of table 5 from the general trend in the variation of β_2 in this region. We are thus left with *only one* unknown parameter ϵ_l. In column 9 of table 6 we give the calculated $(\mathrm{H.F.})_l$ values with the best fit values of ϵ_l. *In order to show that the results are not particularly dependent on the individual variations of ϵ_l with A and Z,*

Table 2

Parent nucleus	r (10^{-13} cm)	z $(=(r-\bar{R})/b)$	$f(z)$ (§5)	$u_0(r)$ (MeV) (equation (1))	$u_2(r)\beta_2$ (MeV) (equation (3.1))	$v_l^{(0)}(r)$ (MeV) (equation (2))	Δv_l (MeV) (equation (8.3))	$W_2(r)\bar{\epsilon}$ (MeV) (equation (8.5))
	9·187	0·182	0·0045	28·839	2·846	22·306	0·618	48·088
	9·608	0·785	0·2544	27·574	2·487	15·265	1·882	75·770
	10·030	1·387	0·7212	26·415	2·187	8·379	2·226	56·990
^{238}Pu	10·451	1·989	0·9518	25·350	1·933	4·148	1·508	31·819
(l = 8)	10·873	2·591	0·9968	24·367	1·717	1·999	0·793	16·155
	11·294	3·193	1·0000	23·457	1·531	0·959	0·384	8·065
	12·558	4·999	—	21·096	1·114	0·106	0·039	0·995
	13·823	6·806	—	19·183	0·835	0·012	0·004	0·120
	9·254	0·023	$8{\cdot}84\times10^{-6}$	30·497	3·554	22·678	0·257	6·358
	9·545	0·438	0·0563	29·569	3·239	19·477	0·903	25·480
	9·835	0·853	0·3069	28·695	2·961	14·217	1·737	29·800
^{254}Fm	10·125	1·267	0·6399	27·873	2·713	9·328	2·060	24·333
(l = 2)	10·416	1·682	0·8704	27·095	2·492	5·791	1·765	16·738
	10·706	2·097	0·9675	26·360	2·295	3·517	1·242	10·708
	11·578	3·342	1·0000	24·376	1·814	0·772	0·290	2·561
	12·449	4·587	—	22·670	1·460	0·169	0·059	0·595

Table 3

Parent nucleus	r (10^{-13} cm)	u_{eff} (MeV) (equation (8.2))	ΔK_1 (MeV) (equation (8.6))	ΔK_2 (MeV) (equation (8.7))	$-A_{\parallel}$ (MeV)	$A_{\perp}$ (MeV) (see § 5)	$\bar{A}$ (MeV)
	9·187	5·914	−12·507	0·205	17·770	12·998	2·742
	9·608	10·428	− 4·570	5·413	35·605	24·284	4·321
	10·030	15·808	5·948	4·115	32·574	21·163	3·251
^{238}Pu	10·451	19·690	7·352	0·690	21·880	13·662	1·815
(l = 8)	10·873	21·577	4·735	0·040	11·140	6·953	0·921
	11·294	22·115	2·537	0·001	5·260	3·320	0·460
	12·558	20·959	0·298	0·000	0·055	1·130	0·057
	13·823	19·159	0·032	0·000	−0·440	−0·209	0·007
	9·254	7·562	− 0·537	0·000	1·255	0·887	0·173
	9·545	9·189	− 4·266	0·624	10·248	6·166	0·695
	9·835	12·741	− 1·420	2·221	13·812	8·125	0·813
^{254}Fm	10·125	16·485	1·865	1·981	12·898	7·445	0·664
(l = 2)	10·416	19·544	3·187	0·793	10·034	5·703	0·457
	10·706	21·600	2·877	0·177	6·718	3·797	0·292
	11·578	23·310	0·869	0·000	0·949	0·578	0·070
	12·449	22·440	0·197	0·000	0·439	−0·196	0·016

it is desirable that we should give, side by side, the corresponding values, particularly of (H.F.)$_8$ *and* (H.F.)$_6$, *for a fixed* ϵ_8 *and* ϵ_6 *respectively* (excepting ^{230}Th, cf. column 8 table 6). The results clearly show that the probable dependence of ϵ_l on A and Z, for large l, is largely masked by the dependence on alpha angular momentum. It is

Table 4

$l = 8$, $E_l = 4{\cdot}996$ MeV		$l = 6$, $E_l = 5{\cdot}195$ MeV		$l = 4$, $E_l = 5{\cdot}352$ MeV		$l = 2$, $E_l = 5{\cdot}452$ MeV	
Nonlocal parameter ϵ_l	Theoretical hindrance (H.F.)$_l$ (equation (16))	ϵ_l	Theoretical (H.F.)$_l$	ϵ_l	Theoretical (H.F.)$_l$	ϵ_l	Theoretical (H.F.)$_l$
0·26	99679	0·20	3052	0·13	421·2	0·10	16·21
0·305	31091	0·22	1613·9	0·15	169·7	0·12	8·00
0·32	22679	0·24	899·7	0·165	95·35	0·14	4·36
0·335	17024	0·26	537·5	0·18	59·40	0·16	2·89
0·35	12766	0·28	347·7	0·205	30·92	0·18	2·09
0·38	8066	0·30	241·7	0·24	15·58	0·20	1·59

Table 5

Parent nucleus	l	Empirical β_2	ϵ_l	ζ_l (equation (13.5))	a_l (equation (13.1))	$q_{0,0}$	$q_{0,l}$ (equation (13))	$y_0 I_0/\chi$ (equation (14))
^{238}Pu	8	0·251	0·35	0·529 (0·541)	−4·355 (−4·499)	7·354	1·520	0·0439
^{242}Cm	8	0·271	0·39	0·648 (0·668)	−4·965 (−5·165)	12·256	3·367	0·0427
^{230}Th	6	0·197	0·165	0·128	−1·771	1·412	0·095	0·0470
^{236}Pu	6	0·257	0·25	0·284	−2·890	2·464	0·612	0·0447
^{238}Pu	6	0·251	0·28	0·352	−3·353	3·285	1·150	0·0447
^{240}Pu	6	0·263	0·30	0·402	−3·640	3·974	1·661	0·0448
^{242}Cm	6	0·271	0·28	0·356	−3·355	3·289	1·153	0·0437
^{244}Cm	6	0·278	0·26	0·311	−3·059	2·728	0·775	0·0441
^{246}Cf	6	(0·280)	0·29	0·379	−3·469	3·546	1·338	0·0426
^{228}Th	4	0·195	0·24	0·254	−2·781	2·313	1·528	0·0464
^{230}Th	4	0·197	0·22	0·224	−2·583	2·071	1·217	0·0474
^{230}U	4	0·220	0·25	0·281	−2·954	2·561	1·856	0·0456
^{234}U	4	0·233	0·24	0·265	−2·825	2·373	1·607	0·0468
^{238}U	4	0·233	0·20	0·186	−2·205	1·706	0·767	0·0475
^{238}Pu	4	0·251	0·16	0·117	−1·557	1·304	0·306	0·0460
^{242}Cm	4	0·271	0·135	0·085	−1·156	1·156	0·150	0·0453
^{244}Cm	4	0·278	0·12	0·066	−0·920	1·095	0·089	0·0456
^{246}Cf	4	(0·280)	0·16	0·119	−1·486	1·274	0·273	0·0440
^{254}Fm	4	(0·288)	0·18	0·153	−1·783	1·419	0·433	0·0429
^{230}Th	2	0·197	0·22	0·223	−2·583	2·071	2·519	0·0483
^{238}U	2	0·233	0·21	0·203	−2·352	1·834	2·071	0·0483
^{238}Pu	2	0·251	0·20	0·183	−2·153	1·665	1·745	0·0465
^{242}Cm	2	0·271	0·20	0·186	−2·140	1·655	1·726	0·0457
^{246}Cf	2	(0·280)	0·175	0·142	−1·711	1·380	1·172	0·0448
^{250}Cf	2	(0·282)	0·16	0·119	−1·499	1·279	0·956	0·0454
^{254}Fm	2	(0·288)	0·15	0·105	−1·320	1·209	0·795	0·0440

therefore not unexpected that for lower l a *fixed* value of ϵ_4 or of ϵ_2 for all nuclei is not available for close agreement with experimental values. One may, however, choose to find the *least* variations of ϵ_4 and of ϵ_2 that may be acceptable. This has led us to classify the nuclei in *two* separate groups, with Th and U isotopes in one group (which may be thought to be in the border region of spherical shape and stable deformations of nuclei) and the rest in the other group, each group being assigned a common ϵ_4 and ϵ_2 respectively. As may be seen from column 8 of table 6, it is interesting that agreements are still quite close except for Cm isotopes. Now we present the calculated values in the tables 1 to 6. Table 5 lists preliminary calculations for use in column 9, table 6.

Table 6

Parent nucleus	l	Alpha energy E_l (MeV)	Empirical β_2	Average ϵ_l	Best-fit values of ϵ_l	Theoretical (H.F.)$_l$: Previous theory (with static potential)[a]	Theoretical (H.F.)$_l$: Present work with ϵ_l as in col. 5	Theoretical (H.F.)$_l$: Present work with ϵ_l as in col. 6	Experimental (H.F.)$_l$
^{238}Pu	8	4·996	0·251	0·37[c]	0·35	28100	9271	12766	12000
^{242}Cm	8	5·596	0·271		0·39	33200	7966	5992	5100
^{230}Th	6	4·436	0·197	—	0·165	9300	—	11482	13000
^{236}Pu	6	5·442	0·257		0·25	706	350	695	640
^{238}Pu	6	5·195	0·251		0·28	630	348	348	360
^{240}Pu	6	4·851	0·263	0·28	0·30	?[b]	363	248	250
^{242}Cm	6	5·806	0·271		0·28	413	320	320	350
^{244}Cm	6	5·511	0·278		0·26	542	323	503	480
^{246}Cf	6	6·462	(0·280)		0·29	336	308	252	280
^{228}Th	4	5·208	0·195		0·24	26	14·6	14·6	13
^{230}Th	4	4·471	0·197		0·22	50	13·6	19·3	20
^{230}U	4	5·658	0·220	0·24	0·25	29·2	13·8	11·8	11
^{234}U	4	4·598	0·233		0·24	43·5	14·2	14·2	14
^{238}U	4	4·020	0·233		0·20	73·2	14·5	33·1	30
^{238}Pu	4	5·352	0·251		0·16	170·3	114·4	114·4	116
^{242}Cm	4	5·964	0·271		0·135	838·5	118·0	364·0	390
^{244}Cm	4	5·658	0·278	0·16	0·12	759·0	122·0	936·0	830
^{246}Cf	4	6·613	(0·280)		0·16	161·9	127·0	127·0	120
^{254}Fm	4	7·064	(0·288)		0·18	73·8	125·0	59·9	57
^{230}Th	2	4·615	0·197	0·22	0·22	2·3	1·2	1·2	1·1
^{238}U	2	4·147	0·233		0·21	1·8	1·3	1·4	1·3
^{238}Pu	2	5·452	0·251		0·20	2·0	2·3	1·6	1·5
^{242}Cm	2	6·066	0·271		0·20	2·1	2·3	1·6	1·7
^{246}Cf	2	6·711	(0·280)	0·175	0·175	2·8	2·4	2·4	2·3
^{250}Cf	2	5·980	(0·282)		0·16	3·4	2·3	3·1	2·9
^{254}Fm	2	7·158	(0·288)		0·15	5·2	2·4	3·9	3·9

[a] Shell-model calculations (Poggenburg *et al.* 1969).

[b] (H.F.)$_6$ for this isotope could not be calculated as reduced H.F. is not given.

[c] For (H.F.)$_8$ in column 8, one could get closer agreement than shown if the average ϵ_8 is taken to be 0·378, i.e. up to the third decimal place; which is avoided (cf. § 5).

5. Discussion

It may be seen that the formula (16) contains a centrifugal term quite different from the ordinary one. It may also be pointed out that the matrices $q_{l,0}$ are now functions of l-dependent quantities and involve a negative exponent in the integrand.

In table 1, the values of R_a, $\bar{R}$ and R_{min} are given for all nuclei for which the hindrance factors are calculated. The eccentricity σ and effective deformation parameter $\bar{\alpha}$ (cf. equation (7.1)) are given for each case in the last two columns.

In tables 2 and 3 we list the calculated values of the different potential functions which contribute to the resultant barrier. Values are given to show how these different functions, namely $f(z) = (2g^{(2)}(z)-1)$, $u_0(r)$, $v_l^{(0)}(r)$, Δv_l, $u_2(r)\beta_2$, $W_2(r)\bar{\epsilon}$, ΔK_1 and ΔK_2 vary with r outward in the immediate neighbourhood of the nuclear surface. In the tables and figures $\bar{\epsilon}$ denotes $(\epsilon_l+(5/4\pi)^{1/2}\epsilon_l^2)$ occurring in equation (8.4). The values listed in tables 2 and 3 are only for α angular momentum $l = 8$ and 2 and for the two extreme cases, ^{238}Pu and ^{254}Fm. Since the results for other l values and for other nuclei are exactly similar, it is considered sufficient for reasons of space to present the values only for these two cases.

To visualize the variations with r of these potential functions we have plotted them relating to ^{238}Pu in the figures 1 and 2.

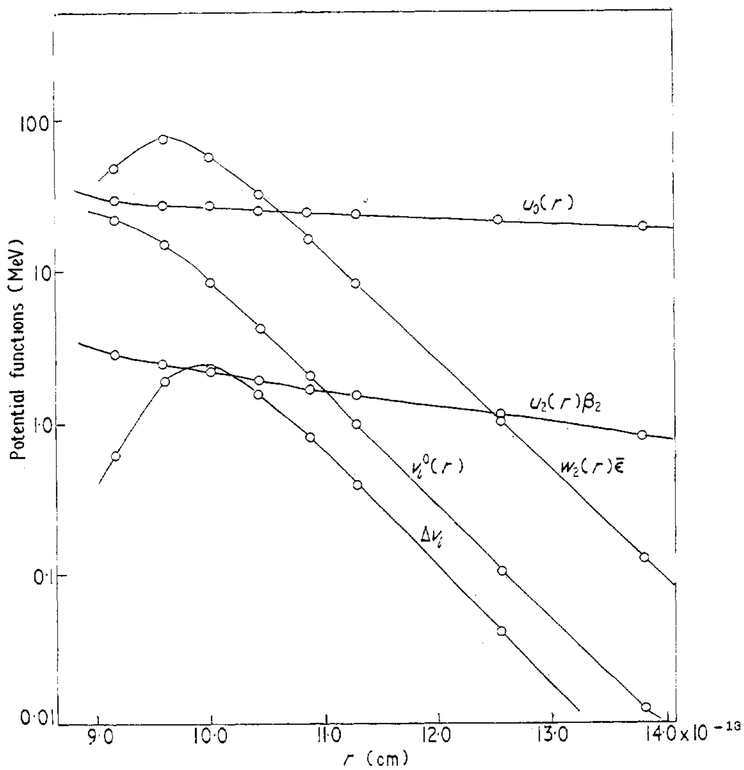

Figure 1. Calculated potential functions in the close neighbourhood of the nuclear surface are plotted against r. The values refer to the ^{238}Pu nucleus and are for α angular momentum $l = 8$. The data are taken from table 2. The different terms represent the electrostatic and the nonlocal potentials, $u_0(r) = 2(Z-2)e^2/r$; $u_2(r)\beta_2$ is its anisotropic part, $v_l^{(0)}$, Δv_l and $W_2(r)\bar{\epsilon}$ are defined in equations (2), (8.3) and (8.5) respectively, where $\bar{\epsilon}$ denotes $(\epsilon_l+(5/4\pi)^{1/2}\epsilon_l^2)$.

Now, for comparison purposes in figure 2, we have reinserted $u_0(r)$ as the reference line. The curve u_{eff} (in figure 2) shows the spherically symmetric potential which is actually effective in the neighbourhood of the nuclear surface and replaces the traditional isotropic electrostatic potential $u_0(r)$ in this region. It is interesting

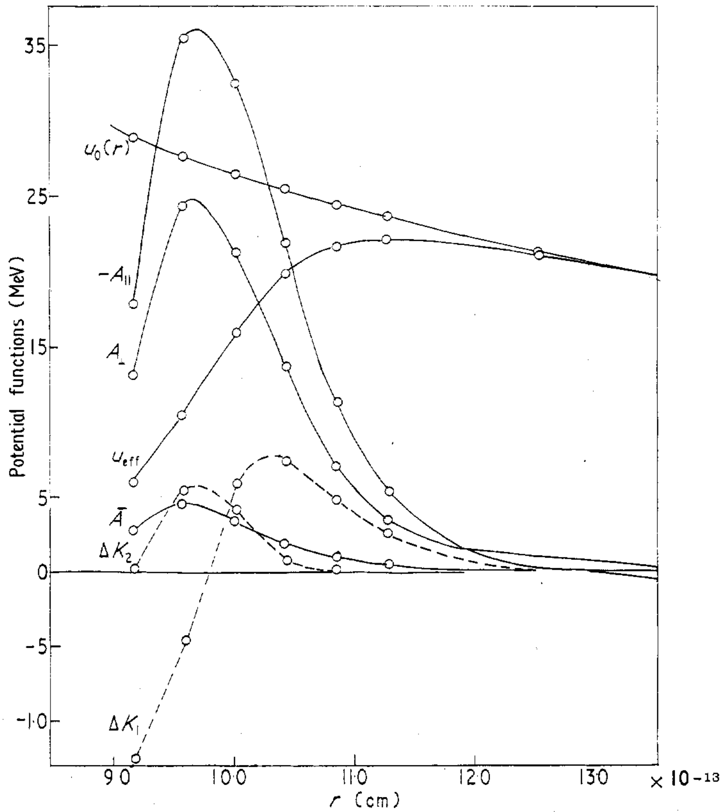

Figure 2. The curves represent the different potential terms as in figure 1. The terms ΔK_1 and ΔK_2 are given by equations (8.6) and (8.7) respectively. The purely anisotropic parts of the nonlocal potential are denoted by $A(\theta) = \mathscr{T}_l(r, \theta)\, 2\mathscr{S}_l(r)$ (see equations (8.4) and (8.1). The functions $A_{\parallel}$, $A_{\perp}$ and $\bar{A}$ correspond to the directions along the nuclear symmetry axis, at right angles to it and along the direction with $\theta = \cos^{-1}(\frac{1}{3})^{1/2}$ respectively. Also see discussion in § 5.

to see from figure 2 how the purely anisotropic part of the barrier denoted in the tables and the figures by $A_\theta = 2\mathscr{S}_l \times \mathscr{T}_l(r, \theta)$ (cf. equations (8.1) and (8.4)) varies with r near the nuclear surface. To visualize how this anisotropic potential A_θ varies along different directions we have calculated A_θ along three principal directions, namely, the direction *parallel* to the nuclear symmetry axis (i.e. for $\theta = 0$ or π), along the normal direction at right angles to the symmetry axis (i.e. for $\theta = \pi/2$), and for the direction with $\theta = \cos^{-1}(\frac{1}{3})^{1/2}$. The A_θ function along these directions is denoted in table 3 and in figure 2 respectively by $A_{\parallel}$, $A_{\perp}$ and $\bar{A}$. Also, since $A_{\parallel}$ is primarily negative, we have plotted $-A_{\parallel}$ to facilitate comparison with the other curves.

In table 4 the calculated values of $(\text{H.F.})_l$ for different α angular momenta and only for the *nucleus* ^{238}Pu are shown for the given value of β_2 and with varying ϵ_l.

It is worth noticing how, for a given l, the theoretical $(\text{H.F.})_l$ changes with ϵ_l. The values are plotted in figure 3.

The curves in figure 3 clearly show that with appropriate values of ϵ_l it is possible to get exact quantitative agreement between the calculated and the empirical values

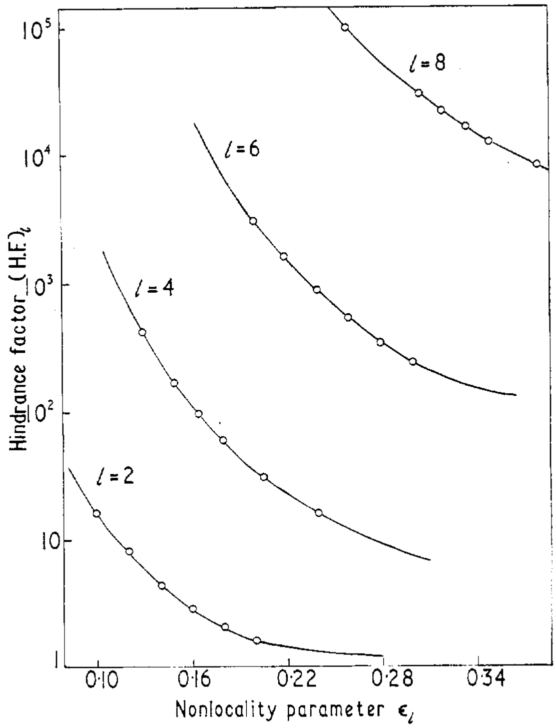

Figure 3. Calculated hindrance factors from equation (16) for different α angular momenta and for the ^{238}Pu nucleus are plotted for the given β_2 and as functions of the nonlocal deformation parameter ϵ_l. Also see § 5.

of $(\text{H.F.})_l$ for all even l values in this case. Similar curves can, of course, be obtained for other nuclei and the same holds good for all the cases to which the present considerations apply. However, it should be mentioned that such agreements may require the values of ϵ_l to be given up to the third or the fourth place of decimals. In view of the approximations used and the fact that the data for β_2 are known up to the second place of decimals, it would be unrealistic to claim such accuracies in the values of ϵ_l. We therefore list in table 6 the chosen values of ϵ_l correct up to the second place of decimals—except for three α groups (namely, $l = 2$ for ^{246}Cf, $l = 4$ for ^{242}Cm and $l = 6$ in ^{230}Th) for which the calculated $(\text{H.F.})_l$ is very sensitive to ϵ_l; here we have given the averages of their lower and upper limits of ϵ_l.

Furthermore it should be mentioned that for $l = 8$ we have given in columns 5 and 6 of table 5 the values of both ζ_8, a_8 and the corresponding ζ_0, a_0, the latter being shown in parentheses immediately below the respective ζ_8 and a_8. It is necessary that, for $l = 8$, $q_{0,0}$ and $q_{0,8}$ be calculated as functions of ζ_0, a_0 and ζ_8, a_8 respectively, strictly in accordance with equations (13.1) and (13.5). This is because ζ_8 differs from ζ_0 and a_8 from a_0 considerably (α energy being normalized for ζ_0 and a_0 in accordance with the definition of $(\mathrm{H.F.})_l$). On the other hand, for $l = 6$, ζ_6, a_6 are nearly the same as ζ_0, a_0 respectively for the same α energy E_6. The differences between ζ_l and ζ_0 and those between a_l and a_0 are still less for $l < 6$. Hence for $l \leqslant 6$ it would be sufficiently accurate to take the same values of ζ_l, a_l in calculating both $q_{0,0}$ and $q_{0,l}$. This is, however, not desirable for $l = 8$ as in that case an avoidable error of about 25% in the values of $(\mathrm{H.F.})_l$ will occur. In this connection it may be mentioned that the empirical values of $(\mathrm{H.F.})_8$ for $^{238}\mathrm{Pu}$, and $(\mathrm{H.F.})_6$ and $(\mathrm{H.F.})_4$ for $^{230}\mathrm{Th}$, given by Perlman and Rasmussen (1957), differ considerably from the values obtained later by Hanna (1959). We have given in table 6 the values of ϵ_l corresponding to the later values shown in the column 10 of the table.

Finally, calculations of $(\mathrm{H.F.})_2$ are given for all relevant elements with $90 \leqslant Z \leqslant 100$, but not for all isotopes—because this is unnecessary as the values of $(\mathrm{H.F.})_2$ are nearly the same for a given Z.

6. Conclusions

From the results given in table 6, one can summarize the trend of variations of ϵ_l with l, z and A and one would come to the following conclusions.

(i) In the first place the nonlocality parameter ϵ_l, as determined by the empirical values of $(\mathrm{H.F.})_l$, is found to increase systematically with increasing l if $l \geqslant 4$ (the case of $l = 2$ is discussed below). As seen from table 6, ϵ_l also varies with Z and A. But this latter variation being small, the *ranges* of ϵ_l for $l = 8$, 6 and 4 remain *discrete.* For example, ϵ_8 is found to lie between 0·40 and 0·35, whereas ϵ_6 varies between 0·30 and 0·25 (except for the unusual case of $^{230}\mathrm{Th}$ with $(\mathrm{H.F.})_6 = 13\,000$, for which $\epsilon_6 = 0{\cdot}165$), and the values of ϵ_4 lie between 0·25 and 0·12. At the same time it is also plausible that ϵ_l is more sensitive to l than to Z and A; the latter variations are not only small but also of the same order as those of the other parameter, namely $\beta_{2,0}$. It may also be noticed that, for $l = 8$ and 6, ϵ_l differs only in the second place of decimals, whereas, for $l \leqslant 4$, *apparently* there is wider variation of ϵ_l with Z and A than normally expected.

(ii) From the striking agreement obtained here between the theoretical and empirical $(\mathrm{H.F.})_l$ values, and also from the consideration that the nonlocal effects are likely to be more predominant for higher values of l, one is led to conclude that the values of ϵ_8 and ϵ_6 given in table 6 are sufficiently accurate and the effects other than those due to nonlocality would probably cause only minor changes of ϵ_8 and ϵ_6.

(iii) One may think that the α groups with $l = 4$ would probably represent the marginal case, so far as the relative importance of the effects of the momentum-dependent potential and the other effects considered in earlier works are concerned. This may be suggested because the values of ϵ_4, as already mentioned, show somewhat wider variations with Z and A than normally expected from the findings for ϵ_8 and ϵ_6. Apparently this straggling of ϵ_4 may be considered as indicative of the existence of other influences not taken into account in the present theory. But on a closer scrutiny the nuclei appear to separate out in two groups. For the *first* group, with $90 \leqslant Z < 94$, ϵ_4 varies from 0·20 to 0·25, whereas for the *second* group of nuclei, with $94 \leqslant Z \leqslant 100$, ϵ_4 ranges from 0·12 to 0·18 with the minimum at $Z = 96$.

It is worth pointing out that α emissions with $l = 8$ and 6 are known to occur (with one exception) only for the second group of nuclei, and for them ϵ_6 lies in the range 0·25 to 0·30. Thus the order of variation of ϵ_4 with Z and A is the same as that for ϵ_6. Besides, the range of ϵ_4 (i.e. from 0·12 to 0·18) is lower than that of ϵ_6 by about as much as the latter is below the range of ϵ_8. Hence the values of ϵ_4 given in table 6 are consistent. It is, therefore, not possible at present to say to what extent the effects, other than those due to nonlocality, would be significant in controlling the hindrance factors for transitions with α angular momentum $l = 4$.

(iv) For $l = 2$, however, there are unmistakable indications of the existence of various influences controlling the α decay, because one finds from table 6 that ϵ_2 not only shows wider variations with Z and A but also turns out to be in general higher than what would be expected from the trend of variations of ϵ_l with l, Z and A. As stated in the previous subsection (iii), the values of ϵ_4 lie in the range from 0·12 to 0·18 and hence ϵ_2 should not be higher than 0·12. On the other hand ϵ_2 is found to be in the range 0·15 to 0·22. It is also to be noticed that, for almost all nuclei, $\epsilon_2 > \epsilon_4$. It therefore appears that for alpha angular momentum $l = 2$ there is some mixing of all the different influences, and the fact that the values of ϵ_2 come out to be higher than expected may be interpreted as a sort of compensation for the inaccuracies in the values of ϵ_2 due to neglect of the other effects (cf. § 1) in the present discussion.

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